CERN-TH/2001-077
Edinburgh 2001-01
NBI-HE-01-04

The C-Theorem and Chiral Symmetry Breaking in Asymptotically Free Vectorlike Gauge Theories

R.D. Ball*

and

P.H. Damgaard†

Theory Division
CERN
CH-1211 Geneva 23
Switzerland

April 3, 2001

Abstract

We confront Cardy's suggested c-function for four-dimensional field theories with the spontaneous breaking of chiral symmetries in asymptotically free vectorlike gauge theories with fermions transforming according to different representations under the gauge group. Assuming that the infrared limit of the c-function is determined by the dimension of the associated Goldstone manifold, we find that this c-function always decreases between the ultraviolet and infrared fixed points.

*On leave from: Department of Physics and Astronomy, University of Edinburgh, Mayfield Road, Edinburgh EH9 3JZ, Scotland.

†On leave from: The Niels Bohr Institute, Blegdamsvej 17, DK-2100 Copenhagen, Denmark.

The C-theorem in two dimensions [1] demonstrates the irreversibility of renormalization group flows by construction of a c-function which is proven to decrease montonically along the flow. As such it is an important ingredient in our understanding of nonperturbative field theories. For four dimensional theories Cardy suggested a particular c-function [2] which he likewise conjectured to decrease along renormalization group trajectories. There is some controversy over whether this C-theorem for four-dimensional field theories has been proven [3] or not [4] (see also refs. [5, 6, 7]). Studies of supersymmetric theories with duality symmetries [8, 9] have found that in all cases Cardy's c-function does decrease along the RG flow. For non-supersymmetric gauge theories, so far the only nonperturbative case that has been considered is that of QCD: Cardy in his original paper [2] found that indeed the proposed c-function did decrease towards the infrared when the number of flavors N_f of Dirac fermions was in a range compatible with asymptotic freedom, and chiral symmetry is spontaneously broken.

In this letter we will extend this analysis to consider the c-function for all asymptotically free vectorlike gauge theories with Dirac fermions. Our main assumption will be that Cardy's conjectured form of the c-function may be used all the way towards the infrared, where chiral symmetry may be broken spontaneously, and where at the fixed point the only massless excitations are those of the associated Goldstone manifold. Comparison of the value of the c-function in the ultraviolet and the infrared will then allow us to test the conjectured C-theorem (or conversely put constraints on the allowed pattern of chiral symmetry breaking if we find that for a particular pattern the theorem is violated). We will consider systematically all possible simple compact gauge groups and all possible irreducible representations r of N_f Dirac fermions coupled vectorially to the gauge fields.

The number N_f and the representation r that the fermions carry will only be constrained by the demand that the theory must be asymptotically free. As we exclude from the beginning the presence of fundamental scalars, the one-loop β-function takes the form

$$\beta(g) = -\frac{g^3}{16\pi^2}\left[\frac{11}{3}\ell(\mathcal{G}) - \frac{4}{3}\ell(r)N_f\right] + \ldots \ , \tag{1}$$

where $\ell(\mathcal{G})$ and $\ell(r)$ are the indices of the representations carried by the gauge bosons and the fermions, respectively. These indices ℓ are defined by

$$\ell\delta_{ab} = 2\mathrm{Tr}(T_a T_b) \tag{2}$$

where $\{T_a\}$ are the generators of the group in the particular representation: they are closely related to the Casimirs C

$$C\delta_{ij} = (T^a T^a)_{ij} \tag{3}$$

(see, *e.g.*, ref. [10] for a review). Because we wish to maintain asymptotic freedom we impose the constraint

$$N_f \ < \ \frac{11}{4}\frac{\ell(\mathcal{G})}{\ell(r)} \tag{4}$$

on the number of flavors of fermions. It may well be that this bound is too weak, and that both confinement and chiral symmetry breaking are lost for smaller values of N_f (as, for example, when there is a perturbative infrared fixed point in the gauge coupling). But this bound must at least always be satisfied. We take all N_f fermions to be strictly massless.

Cardy's proposal for a c-function in four dimensions is based on the Euler term in the trace of the energy-momentum tensor. In the natural normalization where c is unity for one massless scalar degree of freedom [2], it takes the form [11]

$$c \ = \ N_0 + 11N_{1/2} + 62N_1 \tag{5}$$

at the fixed points. Here N_0 counts the number of massless real scalars, $N_{1/2}$ the number of massless Dirac fermions, and N_1 the number of massless vector bosons. Let next $d(\mathcal{G})$ denote the dimension of the gauge group $\mathcal{G}$, and $d(r)$ the dimension of the irreducible representation r. The idea is now quite simple: we compare the value of c in the ultraviolet, where for an asymptotically free gauge theory we have the "fundamental" gauge and fermion degrees of freedom, and thus

$$c_{UV} \;=\; 11d(r)N_f + 62d(\mathcal{G}) \;, \tag{6}$$

with the value in the infrared, where the only massless degrees of freedom are assumed to be those of the Goldstone bosons which arise from chiral symmetry breaking. If $d(G/H)$ is the dimension of the Goldstone manifold G/H, this implies

$$c_{IR} \;=\; d(G/H) \;. \tag{7}$$

If the C-theorem holds, we should find that $c_{UV} \geq c_{IR}$.

To determine the number of Goldstone bosons we need to know the pattern of chiral symmetry breaking $G \to H$ in strongly coupled (confining) vectorlike gauge theories, when N_f (Dirac) fermions transform according to an irreducible representation r of the gauge group $\mathcal{G}$. There are three generic classes of breaking to consider:[1]

- The fermion representation r is pseudo-real: chiral symmetries are enhanced from $SU(N_f) \times SU(N_f)$ to $SU(2N_f)$, and the expected symmetry breaking pattern is $SU(2N_f) \to Sp(2N_f)$.
- The fermion representation r is complex: the expected symmetry breaking pattern is $SU(N_f) \times SU(N_f) \to SU(N_f)$.
- The fermion representation r is real: chiral symmetries are again enhanced to $SU(2N_f)$, and the expected symmetry breaking pattern is $SU(2N_f) \to SO(2N_f)$.

Because of a connection between the Dyson classification in Random Matrix Theory and the three symmetric spaces G/H discussed above [13, 14], it has become customary to label these three symmetry breaking patterns by their Dyson indices $\beta = 1, 2$ and 4, respectively. Of course, there can be exceptions to these scenarios, when for instance the gauge theories contain fundamental scalars, or additional symmetries (such as supersymmetry). We will not consider such cases in this paper.

It is not easy to prove that these symmetry breaking patterns actually do occur dynamically. Two specific cases have been proven by Coleman and Witten [15]: the symmetry breaking pattern of fermions in the defining representations of gauge groups $SU(N_c)$ or $SO(N_c)$ in the large-N_c limit (cases $\beta = 2$ and $\beta = 4$, respectively). The one class that is being left out in the Coleman-Witten paper is the one for which the fermions transform as a pseudo-real representation of the gauge group. This can be achieved by considering gauge groups $Sp(2N_c)$. Taking the large-N_c limit, we easily extend the arguments of ref. [15], and thus prove that indeed the chiral $SU(2N_f)$ symmetry in that case breaks down to $Sp(2N_f)$, as expected from the above classification. Anomaly matching conditions [16] can also be used to constrain the form of vacuum condensates [17].

For our purposes, however, it is sufficient to note that the three generic symmetry breaking patterns each assume maximal breaking of chiral symmetry consistent with the preservation of maximal flavor symmetry. The spontaneous breaking of flavor symmetries in vectorlike gauge theories is prohibited by the Vafa-Witten theorem [18]. It follows that each pattern gives an upper bound on the number of Goldstone bosons in the broken theory, and thus an upper bound on the c-function in the infrared. Thus if the C-theorem is satisfied with maximal breaking, it will be satisfied in general.

[1] For an excellent exposition of these ideas, see *e.g.* ref. [12].

The counting is thus very straightforward in the infrared. For the three different classes of maximal breaking we have:

$$c_{IR} = \begin{cases} d(SU(2N_f)/Sp(2N_f)) \\ d(SU(N_f) \times SU(N_f)/SU(N_f)) \\ d(SU(2N_f)/SO(2N_f)) \end{cases} = \begin{cases} N_f(2N_f-1)-1 & \text{for } \beta=1, \\ N_f^2-1 & \text{for } \beta=2, \\ N_f(2N_f+1)-1 & \text{for } \beta=4. \end{cases} \tag{8}$$

The value taken by Cardy's c-function in the infrared is thus bounded above by a function of N_f and the value of β. Note that in the special case $N_f = 1$, for the symmetry breaking classes of $\beta = 1, 2$, the Goldstone manifold is zero-dimensional. There are then no Goldstone bosons at all, and the only breaking of chiral symmetry is the explicit one due to the anomaly. The theory then has a mass gap in the infrared, and it is consistent to set $c_{IR} = 0$ in those cases. For the $\beta = 4$ class the Goldstone manifold is non-trivial even for $N_f = 1$, however, consistent with the fact that this case is equivalent to having two Majorana fermions.

It is easy to show that

$$N_f^2 - 1 \;\leq\; N_f(2N_f-1)-1 \;<\; N_f(2N_f+1)-1, \qquad \text{for all} \qquad N_f \geq 1, \tag{9}$$

so that for a given N_f the tightest C-theorem constraints are found when the fermions are in real representations r, i.e. when $\beta = 4$. We can use this observation to prove the following lemma:

Lemma: if $c_{UV} \geq c_{IR}$ for fermions in a real representation r_0 of $\mathcal{G}$ with dimension $d(r_0)$ and index $\ell(r_0)$, it will also hold for all other representations r of $\mathcal{G}$ with dimension $d(r) \geq d(r_0)$ and index $\ell(r) \geq \ell(r_0)$.

The proof is straightforward: $d(r) \geq d(r_0)$ means that c_{UV} is greater for r than for r_0, while $\ell(r) \geq \ell(r_0)$ means that the condition (4) on N_f from asymptotic freedom is more relaxed. If r is not real but pseudo-real or complex, c_{IR} is reduced for a given N_f.

Given that the adjoint representation is always real, confirming the C-theorem for the adjoint is thus sufficient to confirm it for all representations except for a finite number of low dimension. In general these must then be considered on a case by case basis. We will be systematic, and go through all the simple compact Lie groups in the Cartan classification, beginning with the orthogonal and symplectic groups, through the unitary groups to the exceptional groups.

$SO(N_c)$**:** Although from the group-theoretical perspective the $SO(N_c)$ groups are very different depending on whether N_c is even or odd, here we may consider all the orthogonal groups together. We take $N_c > 6$, since $SO(2)$ is abelian, and $SO(3) \sim SU(2)$, $SO(5) \sim Sp(4)$ and $SO(6) \sim SU(4)$ will all be treated below. The dimension of $SO(N_c)$ is $d(\mathcal{G}) = N_c(N_c-1)/2$, and the index of the adjoint representation is $\ell(SO(N_c)) = N_c - 2$, so the condition of asymptotic freedom becomes

$$N_f \;<\; \frac{11}{4}\frac{N_c-2}{\ell(r)} \tag{10}$$

First consider the defining representations of $SO(N_c)$: these are all real, so $\beta = 4$, and furthermore have $\ell(r) = 1$ and $d(r) = N_c$. In the ultraviolet we thus have

$$c_{UV} \;=\; 11N_cN_f + 31N_c(N_c-1), \tag{11}$$

while in the infrared

$$c_{IR} \;=\; N_f(2N_f+1)-1\ . \tag{12}$$

The condition $c_{UV} \geq c_{IR}$ thus becomes

$$N_f \;\leq\; \frac{11}{4}(N_c-2)+\frac{21}{4}+\frac{3}{4}\sqrt{41N_c^2-30N_c+1} \tag{13}$$

which is automatically satisfied since the condition of asymptotic freedom (10) in this case gives

$$N_f \;<\; \frac{11}{4}(N_c-2)\ . \tag{14}$$

For all other representations r of $SO(N_c)$ we can use the lemma, since $\ell(r) \geq 1$ and $d(r) \geq N_c$. So the c-function decreases between the ultraviolet and the infrared for all representations of all $SO(N_c)$ groups.

$Sp(2N_c)$**:** The adjoint representations of $Sp(2N_c)$ have indices $\ell(\mathcal{G}) = 2(N_c+1)$, and $d(\mathcal{G}) = N_c(2N_c+1)$ for these gauge groups. The condition of asymptotic freedom (4) thus becomes

$$N_f \;<\; \frac{11}{2}\frac{N_c+1}{\ell(r)}\ . \tag{15}$$

All representations of $Sp(2N_c)$ are either real or pseudo-real, and the fundamental representations of $Sp(2N_c)$ are always pseudo-real. Thus for the fundamental representations

$$c_{UV} \;=\; 22N_cN_f+62N_c(2N_c+1)\ , \tag{16}$$

while

$$c_{IR} \;=\; N_f(2N_f-1)-1\ . \tag{17}$$

The condition $c_{UV} \geq c_{IR}$ then translates into

$$N_f \;\leq\; \frac{11}{2}N_c+\frac{11}{4}+\frac{3}{4}\sqrt{164N_c^2+60N_c+1}\ , \tag{18}$$

while the asymptotic freedom condition for the fundamental representation requires

$$N_f \;<\; \frac{11}{2}(N_c+1)\ . \tag{19}$$

One sees that the inequality (18) is satisfied for all N_c. Because we again here have $\ell(r) = 1$ for the fundamental representation, and all other pseudoreal representations have both larger $\ell(r)$ and larger dimensions $d(r)$, we conclude using an argument similar to that used for the lemma that the c-function decreases from the ultraviolet to the infrared for all pseudo-real representations of $Sp(2N_c)$.

We next turn to the real representations of $Sp(2N_c)$. The smallest real representation is not the adjoint, but rather a representation with dimension $d(r) = N_c(2N_c-1)-1$ and index $\ell(r) = 2(N_c-1)$. For this represention the condition $c_{UV} \geq c_{IR}$ becomes

$$N_f \;\leq\; \frac{1}{4}\left[11N_c(2N_c-1)-10+\sqrt{(11N_c(2N_c-1)-10)^2+8(62N_c(2N_c+1)+1)}\right]\ , \tag{20}$$

which is easily satisfied for all $N_c \geq 2$ since the asymptotic freedom condition is now

$$N_f \;<\; \frac{11}{4}\frac{N_c+1}{N_c-1}\ . \tag{21}$$

Since all other real representations have both larger $\ell(r)$ and larger dimensions $d(r)$, and combining this with our result above for all pseudo-real representations of $Sp(2N_c)$, we conclude that the C-theorem is satisfied for all representations of $Sp(2N_c)$.

$SU(N_c)$**:** We begin with gauge groups $SU(N_c), N_c \geq 3$. The fundamental representations are all complex, they have the common index $\ell(r) = 1$, and the symmetry breaking class is the one of $\beta = 2$. The condition $c_{UV} \geq c_{IR}$ gives

$$N_f \;\leq\; \frac{11}{2}N_c + \sqrt{121N_c^2/4 + 62N_c^2 - 61}\;. \tag{22}$$

Because the condition (4) in this case reads $N_f < 11N_c/2$ the condition (22) is, as observed already by Cardy [2], automatically satisfied. Since all other complex representations of $SU(N_c)$ have both dimension and index greater than that of the fundamental, it follows from an argument similar to that used for the lemma that the C-theorem is satisfied for all complex representations of $SU(N_c)$.

But the groups $SU(N_c)$ have, in general, both real and pseudo-real representations too. Because $d(r)$ is not a monotonic function of $\ell(r)$, it is not possible to select the "lowest" real or pseudo-real representations, check those, and then conclude that all other real or pseudo-real representations will yield a decreasing c-function too. We can, however, consider all the adjoint representations of $SU(N_c)$. They are real, and we therefore have

$$c_{UV} \;=\; (N_c^2 - 1)(11N_f + 62)\;, \tag{23}$$

while

$$c_{IR} \;=\; N_f(2N_f + 1) - 1\;, \tag{24}$$

which leads to

$$N_f \;\leq\; \frac{1}{4}\left[11N_c - 12 + \sqrt{(11N_c - 12)^2 + 8(62N_c^2 - 61)}\right]\;, \tag{25}$$

a bound which is very much above the requirement of $N_f < 11/4$ from asymptotic freedom. So for all adjoint representations of $SU(N_c)$ we have $c_{UV} > c_{IR}$. By the lemma, the same will be true of all representations with dimension and index greater than those of the adjoint.

For the rest, it seems that the only solution is to consider the remaining real and pseudo-real representations of the $SU(N_c)$ groups on a case-by-case basis. We have done this for all non-Abelian gauge groups of rank less than 9, based on the tables of ref. [19]. For $SU(3)$, $SU(5)$, $SU(7)$ $SU(8)$ and $SU(9)$ it turns out that there are no such representations which also satisfy the asymptotic freedom constraint and have smaller dimension and index than the adjoint. The group $SU(4)$ has no pseudo-real representations, and three real representations with $\ell(r) < 22$, as required from asymptotic freedom. The one with lowest index ($\ell(r) = 2$) has dimension 6, which gives

$$c_{UV} \;=\; 66N_f + 930\;\;, \qquad c_{IR} \;=\; N_f(2N_f + 1) - 1\;, \tag{26}$$

and hence a bound of $N_f \geq (65+3\sqrt{1297})/4$, which is much above the bound $N_f < 11$ from asymptotic freedom. Because the two other real representations have both larger indices and larger dimensions, they are even further away from the bound imposed by asymptotic freedom. $SU(6)$ has one relevant representation of index $\ell(r) = 6$, which is pseudo-real and of dimension 20. One easily checks that $c_{UV} > c_{IR}$.

Finally we treat the special case of the gauge group $SU(2)$. Here the condition for asymptotic freedom (4) becomes $\ell(r)N_f < 11$. The fundamental representation is pseudo-real, and the symmetry breaking class is the one of $\beta = 1$. We thus have

$$c_{UV} \;=\; 22N_f + 186\;\;, \qquad c_{IR} \;=\; N_f(2N_f - 1) - 1\;, \tag{27}$$

and hence the condition $N_f \leq 17$, which well encompasses the bound from asymptotic freedom of $N_f < 11$. By a similar argument to that used for the lemma, this implies that $c_{UV} > c_{IR}$ for all other pseudo-real representations of $SU(2)$. It remains to check only the real representations, those of integer isospin j, which correspond to the symmetry breaking class of $\beta = 4$. The index of an isospin-j representation in $SU(2)$ is

$$\ell(r) = \frac{2}{3} j(j+1)(2j+1) . \tag{28}$$

Because of the bound from asymptotic freedom of $\ell(r) < 11$ this means that we need only consider the adjoint ($j = 1$) representation for which $\ell(r) = 4$. Here,

$$c_{UV} = 33N_f + 186 \quad , \qquad c_{IR} = N_f(2N_f + 1) - 1 \; , \tag{29}$$

which leads to $N_f < 8 + (3/2)\sqrt{70}$, which again is well above the bound $N_f < 4$ from asymptotic freedom. We conclude that $c_{UV} > c_{IR}$ for all representations of $SU(2)$.

The Exceptional Groups:

For E_6 the adjoint index is $\ell(\mathcal{G}) = 24$, which gives as condition for asymptotic freedom $\ell(r)N_f < 66$. There are then only two relevant representations: the fundamental of index $\ell(r) = 6$ and dimension 27, and the adjoint which has dimension 78. The fundamental representation is complex. In both cases we find that $c_{UV} > c_{IR}$.

In E_7 the adjoint index is $\ell(\mathcal{G}) = 36$, and the condition of asymptotic freedom is thus $\ell(r)N_f < 99$. There are then two relevant representations, the fundamental of $\ell(r) = 12$ and dimension 56, and the adjoint which has dimension 36. The fundamental representation is pseudo-real. In both cases we again find $c_{UV} > c_{IR}$.

The exceptional group E_8 is special in that the fundamental representation coincides with the adjoint. By the lemma this is the only relevant representation: it has $d(\mathcal{G}) = 248$ and $\ell(\mathcal{G}) = 60$, so the condition of asymptotic freedom is $N_f < 11/4$. It is easy to check that $c_{UV} > c_{IR}$.

For gauge group F_4 the adjoint index is $\ell(\mathcal{G}) = 18$, which gives $\ell(r)N_f < 99/2$ from the condition of asymptotic freedom. There are two relevant representations, the fundamental of index $\ell(r) = 6$ and dimension 26, which is real, and the adjoint of dimension 52. By the lemma, only the fundamental need be checked, and we get $c_{UV} > c_{IR}$.

Finally, the group G_2 has adjoint index $\ell(\mathcal{G}) = 2$, which leads to $\ell(r)N_f < 22$ from the demand of asymptotic freedom. There are then three relevant representations, all real: the fundamental of $\ell(r) = 2$ and dimension 7, the adjoint of dimension 14, and an $\ell(r) = 18$ representation of dimension 27. Again by the lemma it is only necessary to check the fundamental, where $c_{UV} > c_{IR}$.

In conclusion, we have undertaken a systematic examination of Cardy's proposed c-function in the context of asymptotically free vectorlike gauge theories. Our main assumption has been that we can follow the renormalization group flow from the ultraviolet gauge theory degrees of freedom nonperturbatively to an infrared fixed point where the only massless excitations are those of the Goldstone bosons. We also assume that the pattern of symmetry breaking is consistent with the Vafa-Witten theorem, and that there are no exotic phenomena such as exactly massless bound-state fermions in the infrared theory. We have found that the c-function decreases along the RG trajectories in all cases considered: for all gauge groups, with fermions in any representation of the gauge group, and for all allowed patterns of symmetry breaking. Turning this around, we find that a C-theorem would impose no new constraints on the pattern of spontaneous chiral symmetry breaking in asymptotically free vectorlike gauge theories.

It may be of interest to apply the constraints imposed by the C-theorem on theories in which the above assumptions no longer hold. In particular, perhaps we can learn more about the possible symmetry breaking patterns in chiral (but anomaly-free) gauge theories in this way: for chiral theories the Vafa-Witten theorem no longer holds and the pattern of spontaneous symmetry breaking may be more involved. It may also be of interest to consider constraints on chiral symmetry breaking imposed by other criteria, such as the recent proposal based on a decreasing free energy [20].

Acknowledgments
The work of R.D.B. and P.H.D. was supported in part by EU TMR grant nos. ERBFMRXCT97-0122 and ERBFMRXCT98-0194 respectively.

References

[1] A. B. Zamolodchikov, JETP Lett. **43** (1986) 730.

[2] J. L. Cardy, Phys. Lett. B **215** (1988) 749.

[3] S. Forte and J. I. Latorre, Nucl. Phys. B **535** (1998) 709 [hep-th/9805015].

[4] H. Osborn and G. M. Shore, Nucl. Phys. B **571** (2000) 287 [hep-th/9909043].

[5] I. Jack and H. Osborn, Nucl. Phys. B **343** (1990) 647.

[6] A. Cappelli, D. Friedan and J. I. Latorre, Nucl. Phys. B **352** (1991) 616.

[7] A. Cappelli, G. D'Appollonio, R. Guida and N. Magnoli, hep-th/0009119;
A. Cappelli, R. Guida and N. Magnoli, hep-th/0103237.

[8] F. Bastianelli, Phys. Lett. B **369** (1996) 249 [hep-th/9511065].

[9] D. Anselmi, D. Z. Freedman, M. T. Grisaru and A. A. Johansen, Nucl. Phys. B **526** (1998) 543 [hep-th/9708042].
D. Anselmi, J. Erlich, D. Z. Freedman and A. A. Johansen, Phys. Rev. D **57** (1998) 7570 [hep-th/9711035].

[10] R. Slansky, Phys. Rept. **79** (1981) 1.

[11] S. M. Christensen and M. J. Duff, Nucl. Phys. B **154** (1979) 301.

[12] M. E. Peskin, Nucl. Phys. B **175** (1980) 197.

[13] M.R. Zirnbauer, J. Math. Phys. **37** (1996) 4986.

[14] J. Verbaarschot, Phys. Rev. Lett. **72** (1994) 2531 [hep-th/9401059].

[15] S. Coleman and E. Witten, Phys. Rev. Lett. **45** (1980) 100.

[16] G. 't Hooft, *in* Recent Developments in Gauge Theories (G. 't hooft et al., Eds.), Plenum Press, New York, 1980.

[17] Y. I. Kogan, M. A. Shifman and M. I. Vysotsky, Sov. J. Nucl. Phys. **42** (1985) 318.

[18] C. Vafa and E. Witten, Nucl. Phys. B **234** (1984) 173.

[19] W.G. McKay and J. Patera, "Tables of Dimensions, Indices, and Branching Rules for Representations of Simple Lie Algebras", Marcel Dekker, Inc. (New York) 1981.

[20] T. Appelquist, A. G. Cohen and M. Schmaltz, Phys. Rev. D **60** (1999) 045003 [hep-th/9901109].